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TECHNICAL REPORT

Office of Naval Research Contract No. N00014-86-K0029

SYSTEM SIZE AND REMAINING SERVICE IN M/G/1

by

Martin Krakowski

Report No. GMU/22474/114
August 15, 1989

Department of Operations Research and Applied Statistics
School of Information Technology and Engineering
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Copy No. 12

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

System Size and Remaining Service in M/G/1

5. TYPE OF REPORT & PERIOD COVERED

Technical Report

6. PERFORMING ORG. REPORT NUMBER

GMU/22474/113

7. AUTHOR(s)

Martin Krakowski

8. CONTRACT OR GRANT NUMBER(s)

N00014-86-K-0029

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Department of Operations Research and
Applied Statistics
George Mason University, Fairfax, Va. 22030

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

TaskK-B
Project 4118150

11. CONTROLLING OFFICE NAME AND ADDRESS

Office of Naval Research
800 North Quincy Street
Arlington, Va. 22217

12. REPORT DATE

August 14, 1989

13. NUMBER OF PAGES

15

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

applied probability
computational probability
probability
queues
stochastic modeling
stochastic service systems
single-server queues

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(On Following Page)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

SYSTEM SIZE AND REMAINING SERVICE IN M/G/1

Abstract

Wishart [1961] and Takács [1963] derived the joint distribution of the size N and residual service R as encountered by a new arrival into a regular M/G/1. Wishart obtained the following expression for the generating function $\Pi(z,x) = \sum_0^\infty \Pi_j(x) z^j$, where $\Pi_j \stackrel{d}{=} \Pr\{R \le x, N=j\}$:

$$\Pi(z,x) = \frac{(1-\rho)\lambda z(1-z)}{\eta(\lambda-\lambda z)-z} \int_0^\infty e^{-\lambda(1-z)\xi} \{H(\xi+x)-H(\xi)\} d\xi \ .$$

We exploit the fact that the system size N is known and find the conditional r.v.'s $\{R_j$ = residual service seen while $N=j.\}$ Our method seems better suited to numerical work and we extend it to some variants of M/G/1: M/G/1/K, and then M/G/1 and M/G/1/K with state-dependent service and arrival rates.

Notation (for browsers; the symbols are also defined in context)

N = size of system;

N_* = size of system provided server works;

λ = (poissonian) arrival frequency

λ_j = arrival frequency when $N = j$

x = service time;

x_j = (state-dependent) service time of a customer whose service starts when $N = j$;

$\tilde{B}(t) = Pr(x \leq t)$;

$\tilde{B}_j(t) = Pr(x_j \leq t)$;

$P_j = Pr(N = j)$;

$P_* = Pr$(server busy);

$\psi(x)$ is an arbitrary function of x; $\psi(x,y)$ is an arbitrary function of x,y; $D_2\psi(x,y) = \partial\,\psi(x,y) \big/ \partial y$; partial derivative with respect to the second argument

f_{ij} = frequency of jumps "$i \rightarrow j$"

$c{*}x = x_1 + x_2 + + x_c$ where the x_i are free copies of x (i.i.d. r.v.'s equivalent to x and also independent of other variables within the argument)

$\psi(Z)$ is an arbitrary function of Z such that $E\psi(Z) \leq \infty$ (all terms entering analysis are required to have finite expectation, e.g. $E\psi'(Z)$)

$E\psi(Z)$ is called the omni-transform of $\psi(Z)$. When $\psi(Z) = \exp(-sZ)$ and $Z \geq 0$ then $E\psi(Z)$ is the L-S transform of Z

Section 1 Service Residues as Conditioned on System Size in M/G/1

Find R_j, the residual service time ("residue") of the ongoing service as seen when $N = j, j \geq 1$. Assume the $P_j = \Pr(N=j)$ to be known (e.g. Gross and Harris 1985).

Definition Let $\psi(X)$ be an arbitrary function of the process X with finite $E\psi(X)$ and $E\psi'(X)$. We call $E\psi(X)$ the *omni-transform* of X; if $\psi(X) = e^{-sX}$ we get the L-S transform. The *balance of the process* $\psi(X)$ is the equation $Ed\psi(X) = 0$ for a random dt.

The essence of the omni-method is to study the balance of $\psi(X)$ rather than of X. Among the method's advantages are freedom to choose ψ, notational ease in handling sums and mixtures of r.v.'s, and bypassing L-S transforms in many contexts.

Definition $\qquad Z_j \stackrel{d}{=} R_j$ if $N = j$ and $Z_j \stackrel{d}{=} 0$ if $N \neq j$

The process $\psi(Z_j)$ varies by aging and jumps: $j-1 \rightarrow j$, $j+1 \rightarrow j$, $j \rightarrow j-1$ and $j \rightarrow j+1$. We assume $Ed\psi(Z_j) = 0$ for a random dt and work out the balance. For a random dt:

(a) aging adds: $Ed\psi(Z_1)|_{aging} = P_1 E[\psi(R_1 - dt) - \psi(R_1)] = -dt\, P_1 E\psi'(R_1)$; $dR_1 = -dt$

(b) jumps "0→1" add: $Ed\psi(Z_1)|_{0\rightarrow 1} = dt\, f_{01} E[\psi(x) - \psi(0)]$; $f_{01} = \lambda P_0$

(c) jumps "2→1" add: $Ed\psi(Z_1)|_{2\rightarrow 1} = dt\, f_{21} E[\psi(x) - \psi(0)]$; $f_{21} = f_{12} = \lambda P_1$

(d) jumps "1→0" add: $Ed\psi(Z_1)|_{1\rightarrow 0} = dt\, f_{10} E[\psi(0) - \psi(0)] = 0$

(since $Z_1 = R_1 = +0$ just before "1→0" and $Z_1 = 0$ in any state other than "1")

(e) jumps "1→2" add: $Ed\psi(Z_1)|_{1\rightarrow 2} = dt\, f_{12} E[\psi(0) - \psi(R_1)]$; $f_{12} = \lambda P_1$

From (a) through (e) we get the balance of $\psi(Z_1)$, i.e. $Ed\psi(Z_1) = 0$,

$$\boxed{P_1 E\psi'(R_1) + f_{12} E[\psi(R_1) - \psi(0)] = (f_{01} + f_{21}) E[\psi(x) - \psi(0)]} \tag{1.1}$$

where $f_{01} = f_{10} = \lambda P_0$ and $f_{12} = f_{21} = \lambda P_1$. The right side of (1.1) is known.

Definition The *omni-convention* calls for mentally applying the expectation operator E to each side of an omni-equation; in case of ambiguity we retain E. (This convention is kin to summation convention in matrix and tensor calculus.) E.g. (1.1) becomes

$$\boxed{P_1 \psi'(R_1) + f_{12}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x) - \psi(0)]} \tag{1.2}$$

Let us consider the changes in $E\psi(Z_j)$ for a $j \geq 2$. During a random dt:

(A) aging adds: $Ed\psi(Z_j)|_{aging} = P_j E[\psi(R_j - dt) - \psi(R_j)] = -dtP_j E\psi'(R_j)$; $dR_j = -dt$

(B) jumps "$j-1 \to j$" add: $Ed\psi(Z_j)|_{j-1 \to j} = dt f_{j-1,j} E[\psi(R_{j-1}) - \psi(0)]$; $f_{j-1,j} = \lambda P_{j-1}$

(C) jumps "$j+1 \to j$" add: $Ed\psi(Z_j)|_{j+1 \to j} = dt f_{j+1,j} E[\psi(x) - \psi(0)]$; $f_{j+1,j} = f_{j,j+1} = \lambda P_j$

(D) jumps "$j \to j-1$" add: $Ed\psi(Z_j)|_{j \to j_{-1}} = dt f_{j,j-1} E[\psi(0) - \psi(0)] = 0$

(E) jumps "$j \to j+1$" add: $Ed\psi(Z_j)|_{j \to j+1} = dt f_{j,j+1}[\psi(0) - \psi(R_j)]$; $f_{j,j+1} = \lambda P_j$

From (A) through (E) we get the balance of $\psi(Z_j)$

$$\boxed{P_j \psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] = f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x) - \psi(0)]} \tag{1.3}$$

where $f_{j,j-1} = f_{j-1,j} = \lambda_{j-1j} P_{j-1}$ and $f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$. The right side of (1.3) is known when we solve for successive values of $j \geq 1$.

From (1.2) and (1.3) we get equations for moments; or L-S transforms; or tail distributions of R_j by setting $\psi(R_j) = R_j^{\,i}$ for $i \geq 1$; or $\psi(R_j) = \exp(-sR_j)$; or $\psi(R_j) = \xi_j$ where $\xi_j = 1$ if $R_j > t$ and $\xi_j = 0$ if $R_j > t$ for then $\tilde{H}_j(t) \stackrel{d}{=} \Pr(R_j > t) = E\xi_j$.

Note: We can set $E\psi(R_j) = \Pr(R_j > t)$ if we know that $\Pr(R_j > t) = E\xi(R_j)$ for some $\xi(R_j)$. In linear omni-equations with constant coefficients, as in our paper, we can view ψ as a general functional. We need not then find a $\xi(R_j)$ and need no omni-convention.

Let $\psi(R_j) = \tilde{H}_j(t) \stackrel{d}{=} \Pr(R_j > 0)$ in (1.1) and (1.2) and let $\psi(x_j) = \Pr(x_j > t) \stackrel{d}{=} \tilde{B}(t)$. Then

$$\psi'(R_j) = \lim \frac{\psi(R_j - dt) - \psi(R_j)}{-dt} = \lim \frac{\Pr(R_j - dt > t) - \Pr(R_j > t)}{-dt} =$$

$$\lim \frac{\tilde{H}_j(t+dt) - \tilde{H}_j(t)}{-dt} = -\tilde{H}'(t)$$

and we get with $\tilde{B}(t) \overset{d}{=} \Pr(x>t)$

$j=1$ $$\boxed{-P_1 \tilde{H}_1'(t) + \lambda P_1 \tilde{H}_1(t) = (\lambda P_0 + \lambda P_1)\tilde{B}(t)} \qquad (1.4a)$$

$j \geq 2$ $$\boxed{-P_j \tilde{H}_j'(t) + \lambda P_j \tilde{H}_j(t) = \lambda P_{j-1} \tilde{H}_{j-1}(t) + \lambda P_j \tilde{B}(t)} \qquad (1.4b)$$

$\tilde{H}_j(t)$ can be found from (1.4a); from (1.4.b) we can derive $\tilde{H}_j(t)$ if $\tilde{H}_{j-1}(t)$ is known. Thus we can find the $\tilde{H}_j(t)$ for successive j. Equations (1.4) are easily verified for M/M/1 with $\tilde{H}_j(t) = \tilde{B}(t) = e^{-\mu t}$ for each j. Moreover (1.4a) implies

$j \geq 1$ $$\tilde{H}_j'(t) \to 0 \text{ as } t \to \infty, \text{ and } P_j \tilde{H}_j'(t) \to -\lambda P_{j-1} \text{ as } t \to 0 \qquad (1.5)$$

From (1.1a) and (1.2a) we get the recursive relations for $\bar{R}_j$

$$\lambda P_1 \bar{R}_1 = \rho P_0 - (1-\rho) P_1 \quad \text{and} \quad \lambda P_{j+1} \bar{R}_{j+1} = \lambda P_j \bar{R}_j - (1-\rho) P_{j+1} \qquad (1.6)$$

If we know all the conditional $\psi(R_j)$ we can get the $\psi(w)$. Clearly

$$w_0 = 0 \text{ and } w_j = \psi(w|N=j) = \psi((j-1)*x + R_j), \quad j \geq 1 \qquad (1.7)$$

$$\psi(w) = P_0 \psi(0) + P_1 \psi(R_1) + P_2 \psi(1*x + R_2) + P_3 \psi(2*x + R_3) + P_4 \psi(3*x + R_3) + + \qquad (1.8)$$

where $c*x \overset{d}{=} x_1 + + x_c$; the x_i are free copies of x (i.i.d. copies of the generic x and independent of the R_j).

Note: Using the renewal relation (Krakowski 1987)

$$\psi(Z) - \psi(0) = \bar{Z}\, E\psi'(\Re Z), \quad Z \geq 0 \qquad \Re Z = \text{residue of } Z$$

we can recycle (1.2) and (1.3) into (omni-convention still holds!)

$$\boxed{P_1 \psi(R_1) + f_{12} \bar{R}_1 \psi(\Re R_1) = (f_{01} + f_{21}) \bar{x}\, \psi(\Re x)} \qquad (1.9a)$$

$$\boxed{P_j \psi(R_j) + f_{j,j+1} \bar{R}_j \psi(\Re R_j) = f_{j-1,j} \bar{R}_{j-1} \psi(\Re R_{j-1}) + f_{j+1,j} \bar{x}\, \psi(\Re x)} \qquad (1.9b)$$

Equations (1.9) have no derivatives ψ' and thus in a sense are integrals of (1.2) and (1.3). But since both R_j and $\Re R_j$ are arguments in (1.9) there is no labor saved unless we take

a special interest in the $\Re R_j$ in addition to R_j.

$\boxed{\textit{Conjecture}\text{: } R_j \rightarrow \text{residue of } x \text{ as } j \rightarrow \infty}$

Section 2 N, R in M/G/1 with State Dependent Service

We modify M/G/1 as follows. A service which starts when $N=j$ lasts $x_j, j\geq 1$. Our problem is to find the R_j for $j\geq 1$ assuming that the P_j are known (Harris 1967, Gross and Harris 1985, pp.289-292; Krakowski July 1986; a closely related vacation model was treated by Harris & Marchal 1988.)

Let $Z_j \stackrel{d}{=} R_j$ if $N=j$ and $Z_j \stackrel{d}{=} 0$ if $N\neq j$; $\psi(Z_j)$ is arbitrary except for $E\psi(Z_j)<\infty$ and $E\psi'(Z_j)<\infty$. During a random dt

(a) aging adds: $Ed\psi(Z_1)|_{aging} = P_1\, E[\psi(R_1-dt)-\psi(R_1)] = -P_1\, dt\, E\psi'(R_1)$; $dR_1 = -dt$

(b) jumps "0→1" add: $Ed\psi(Z_1)|_{0\to 1} = dt\, f_{01}\, E[\psi(x_1)-\psi(0)]$; $f_{01} = \lambda P_0$

(c) jumps "2→1" add: $Ed\psi(Z_1)|_{2\to 1} = dt\, f_{21}\, E[\psi(x_1)-\psi(0)]$; $f_{21} = f_{12} = \lambda P_1$

(d) jumps "1→0" add: $Ed\psi(Z_1)|_{1\to 0} = dt\, f_{10}\, E[\psi(0)-\psi(0)] = 0$

(e) jumps "1→2" add: $Ed\psi(Z_1)|_{1\to 2} = dt\, f_{12}\, E[\psi(0)-\psi(R_1)]$; $f_{12} = \lambda P_1$

From (a) through (e) we get (mind the omni-convention!)

Balance of $\psi(Z_1)$ $$\boxed{P_1\psi'(R_1)+f_{12}[\psi(R_1)-\psi(0)] = (f_{01}+f_{21})[\psi(x_1)-\psi(0)]} \tag{2.1}$$

where $f_{01} = f_{10} = \lambda P_0$ and $f_{12} = f_{21} = \lambda P_1$. The right side of (2.1) is known.

For $j\geq 2$, during a random dt:

(A) aging adds: $Ed\psi(Z_j)|_{aging} = P_j\, E[\psi(R_j-dt)-\psi(R_j)] = -dt\, P_j\, \psi'(R_j)$; $dR_j = -dt$

(B) jumps "j−1→j" add: $Ed\psi(Z_j)|_{j-1\to j} = dt\, f_{j-1,j}\,[\psi(R_{j-1})-\psi(0)]$; $f_{j-1,j} = \lambda P_{j-1}$

(C) jumps "j+1→j" add: $Ed\psi(Z_j)|_{j+1\to j} = dt f_{j+1,j}\, E[\psi(x_j)-\psi(0)]$; $f_{j+1,j} = f_{j,j+1} = \lambda P_j$

(D) jumps "j→j − 1" add: $Ed\psi(Z_j)|_{j\to j-1} = dt\, f_{j,j-1}[\psi(0)-\psi(0)] = 0$

(E) jumps "j→j + 1" add: $Ed\psi(Z_j)|_{j\to j+1} = dt\, f_{j,j+1}\, E[\psi(0)-\psi(R_j)]$; $f_{j,j+1} = \lambda P_j$

From (A) through (E) we get the balance of $\psi(Z_j)$ (mind the omni-convention!)

$$\boxed{P_j\,\psi'(R_j) + f_{j,j+1}\,[\psi(R_j) - \psi(0)] = f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}\,[\psi(x_j) - \psi(0)]} \tag{2.2}$$

where $f_{j,j-1} = f_{j-1,j} = \lambda_{j-1j} P_{j-1}$ and $f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$. The right side of (2.2) is known when we solve for successive values of $j \geq 1$.

When all service lengths x_j are free copies of a generic x, i.e. $\psi(x_j) = \psi(x)$ for each $j > 1$, then (2.1) and (2.2) become (1.2) and (1.3) respectively - as they should.

We get $\tilde{H}_j(t) \stackrel{d}{=} Pr(R_j > t)$ by setting $E\psi(R_j) = Pr(R_j > t)$.

Note: It is enough to know that $E\xi(R_j) = Pr(R_j > t)$ for some function $\xi(R_j)$ - we do not have to actually determine $\xi((R_j)$.

It follows that

$j = 1$
$$\boxed{P_1\,\tilde{H}_1'(t) - \lambda P_1\,\tilde{H}_1(t) = \lambda P_0 - \lambda(P_0 + P_1)\,\tilde{B}_1(t)} \tag{2.3a}$$

$j \geq 2$
$$\boxed{P_j\,\tilde{H}_j'(t) - \lambda P_j\,\tilde{H}_j(t) = \lambda P_{j-1} - \lambda P_{j-1}\,\tilde{H}_{j-1}(t) - \lambda P_j\,\tilde{B}_j(t)} \tag{2.3b}$$

The right side of (2.3a) is known; so is the right side of (2.3b) for each j when the $\tilde{H}_i(t)$ are known for $i < j$. (2.3a,b) imply that for each $j \geq 1$,

$$P_j\;\tilde{H}_j'(0) = \lambda\,P_{j-1} \tag{2.4}$$

From (2.1) and (2.2) we get

$$\lambda P_1\,\bar{R}_1 = (P_0 + P_1)\,\lambda x_1 - P_1 \quad \textit{and} \quad \lambda P_{j+1}\,\bar{R}_{j+1} = \lambda P_j \bar{R}_j - (1 - \lambda x_{j+1})\,P_{j+1} \tag{2.5}$$

When, for each $j \geq 1$, $\tilde{B}_j(t) = \tilde{B}(t)$, then (2.3a,b) become (1.4a,b), as they should.

$j = 1$
$$\boxed{-P_1\,\tilde{H}_1'(t) + \lambda P_1\,\tilde{H}_1(t) = (\lambda P_0 + \lambda P_1)\,\tilde{B}(t)} \tag{1.4a}$$

$j \geq 2$
$$\boxed{-P_j\,\tilde{H}_j'(t) + \lambda P_j\,\tilde{H}_j(t) = \lambda P_{j-1}\,\tilde{H}_{j-1}(t) + \lambda P_j\,\tilde{B}(t)} \tag{1.4b}$$

The question arises, Can we derive the load or delay from (2.1) and (2.2)? Unfortunately, we see no fair way. In our model with state-dependent service the virtual load and delay (and kindred time lengths) depend on future arrivals; this makes them essentially more complex.

Section 3 The R_j and the Load In M/G/1/K

Consider an M/G/1/K, i.e. where $N \leq K$; customers arriving while $N = K$ are lost. We consider the load (backlog, unfinished work) L; $L = w$, the virtual delay, when $N < K$ but for $N = K$ w is not defined. We assume the P_j to be known (Gross and Harris, p. 279-285, 1985). Clearly

$$\psi(L) = P_0 \psi(0) + P_1 \psi(R_1) + P_2 \psi(1*x + R_2) + P_3 \psi(2*x + R_3) + \ + P_K \psi((K-1)*x + R_3) \quad (3.1)$$

Define now $\psi(Z_j)$ as before: $\psi(Z_j) \stackrel{d}{=} \psi(R_j)$ if $N = j$ and $\psi(Z_j) \stackrel{d}{=} \psi(0)$ otherwise. The balance of $\psi(Z_j)$ clearly yields the same equations for $j = 1$ and for $1 < j < K$ as the balance for regular M/G/1. For $j = K$ during a random dt

(a) aging adds:

$$E d\psi(Z_K)|_{aging} = P_K E[\psi(R_K - dt) - \psi(R_K)] = -dt\, P_K \psi'(R_K);\ dR_K = -dt$$

(b) the jumps "$K-1 \to K$" add:

$$E d\psi(Z_K)|_{K-1 \to K} = dt\, f_{K-1,K} E[\psi(R_{K-1}) - \psi(0)];\ f_{K-1,K} = \lambda P_{K-1}$$

(c) the jumps "$K \to K-1$" add:

$$E d\psi(Z_K)|_{K \to K-1} = dt\, f_{K \to K-1} E[\psi(0) - \psi(0)] = 0$$

Therefore

$j = 1$:
$$P_1 \psi'(R_1) + f_{12}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x) - \psi(0)] \quad (3.2a)$$

$1 < j < K$:
$$P_j \psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] =$$
$$= f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x) - \psi(0)] \quad (3.2b)$$

$j = K$
$$P_K \psi'(R_K) = f_{K-1,K}[\psi(R_{K-1}) - \psi(0)] \quad (3.2c)$$

The tail distributions $\tilde{H}_j(t) \stackrel{d}{=} \Pr(R_j > t)$ satisfy

$j = 1$
$$-P_1 \tilde{H}_1'(t) - \lambda P_1 \tilde{H}_1(t) = (\lambda P_0 + \lambda P_0)\tilde{B}(t) \quad (3.3a)$$

$1 < j < K$
$$-P_j \tilde{H}_{j-1}'(t) + \lambda P_j \tilde{H}_j(t) = \lambda P_{j-1} \tilde{H}_{j-1}(t) + \lambda P_j \tilde{B}(t) \quad (3.3b)$$

j = K $$P_K \tilde{H}'_K(t) = \lambda P_{K-1} \tilde{H}_{K-1}(t) \tag{3.3c}$$

Starting with j=1 we can compute the successive $\tilde{H}_j(t)$. Defining $R=0$ for $N=0$ (readers may favor other definitions) we have

$$\psi(N,R) = P_0\,\psi(0,0) + P_1\,\psi(1,R_1) + P_2\,\psi(2,R_2) + \ + P_K\,\psi(K,R_K) \tag{3.4}$$

For the load L (unfinished work, backlog) we have

$$\psi(L) = P_0\,\psi(0) + P_1\,\psi(R_1) + P_2\,\psi(x + R_2) + P_3\,\psi(2{*}x + R_3) + \ + P_K\psi((K-1){*}x + R_K) \tag{3.5}$$

which appears to be new. The economic motivation must be strong to numerically evaluate the moments or the distribution of L; but the complexity appears to inhere in the problem, not in our method.

Section 4 M/G/1 With State-Dependent Service And Arrival Rates

We extend now M/G/1 with state-dependent service lengths of Section 2 by allowing also state-dependent arrival rates. The service time is still determined at the instance in which a service begins and equals x_i, i being the size of the system, including the customer to be served. The arrival rate is λ_j when system size is N=j. *Let us act as if the P_j are known. So soon someone derives these P_j our derivation of N, R for this model will be completed.*

Let, as in Section 2, $Z_j \overset{d}{=} R_j$ if $N=j$ and $Z_j \overset{d}{=} 0$ if $N \neq j$; $\psi(Z_j)$ is arbitrary but subject to $E\psi(Z_j) < \infty$ and $E\psi'(Z_j) < \infty$.

It is easy to see that equations (2.1a) and (2.2a) stay valid but their frequency coefficients reflect the changed arrival and service rates. Clearly we now have

$$\boxed{P_1 \psi'(R_1) + f_{12}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x_1) - \psi(0)]} \qquad (2.1a)=(4.1)$$

$$\boxed{P_j \psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] = f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x_j) - \psi(0)]} \qquad (2.2a)=(4.2)$$

with $f_{01} = \lambda_0 P_0$; $f_{12} = f_{21} = \lambda_1 P_1$; $f_{j-1,j} = \lambda_{j-1} P_{j-1}$; $f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$.

From (4.1) and (4.2) we get $\tilde{H}_j(t) \overset{d}{=} \Pr(R_j > t)$ by setting $E\psi(R_j) = \Pr(R_j > t)$. Thus

$$j=1 \qquad \boxed{P_1 \tilde{H}_1'(t) - \lambda P_1 \tilde{H}_1(t) = \lambda_0 P_0 - (\lambda_0 P_0 + \lambda_1 P_1)\tilde{B}_1(t)} \qquad (4.3a)$$

$$j \geq 2 \qquad \boxed{-P_j \tilde{H}_j'(t) + \lambda_j P_j \tilde{H}_j(t) = \lambda_{j-1} P_{j-1} \tilde{H}_{j-1}(t) + \lambda_j P_j \tilde{B}_j(t)} \qquad (4.3b)$$

The right side is known for (4.3a), and for (4.3b) if we solve for successive $j \geq 1$. (4.3a) and (4.3b) imply that for each $j \geq 1$

$$\tilde{H}_j'(0) = 0 \qquad (4.4)$$

(4.1) and (4.2) imply

$$\lambda_1 P_1 \bar{R}_1 = (\lambda_0 P_0 + \lambda_1 P_1)\bar{x}_1 - P_1 \quad \textit{and} \quad \lambda_j P_j \bar{R}_j = \lambda_{j-1} P_{j-1,j} \bar{R}_{j-1} + \lambda_j x_j P_j \qquad (4.5)$$

Section 5 M/G/1/K With State-Dependent Service and Arrival Rates

We extend M/G/1/K with state-dependent services of Section 4 by allowing state-dependent arrival rates; service time x_i still depends on system size (including customer served) when service starts. The arrival rate is λ_j when $N=j$. We act as if the P_j are known, but there seems to be no analytical derivation yet available.

Define, as in Section 2, $Z_j \stackrel{d}{=} R_j$ if $N=j$, and $Z_j \stackrel{d}{=} 0$ if $N \neq j$; $j \geq 1$. Let $\psi(Z_j)$ satisfy $E\psi(Z_j) < \infty$, $E\psi'(Z_j) < \infty$, and be otherwise arbitrary; then

$$\psi(Z_j) = (1 - P_j)\psi(0) + P_j\,\psi(R_j) \tag{5.1}$$

It is clear that the equations (3.3a,b,c) are still valid so that

$j=1$ $$P_1\,\psi'(R_1) + f_{12}[\psi(R_1) - \psi(0)] = (f_{01} + f_{21})[\psi(x_1) - \psi(0)] \qquad (3.3a)=(5.2a)$$

$1<j<K$ $$P_j\,\psi'(R_j) + f_{j,j+1}[\psi(R_j) - \psi(0)] = f_{j-1,j}[\psi(R_{j-1}) - \psi(0)] + f_{j+1,j}[\psi(x_j) - \psi(0)] \qquad (3.3b)=(5.2b)$$

$j=K$ $$P_K\,\psi'(R_K) = f_{K-1,K}[\psi(R_{K-1}) - \psi(0)] \qquad (3.3c)=(5.2c)$$

$f_{01} = f_{10} = \lambda_0 P_0$; $f_{12} = f_{21} = \lambda_1 P_1$; $f_{j-1,j} = f_{j,j-1} = \lambda_{j-1} P_{j-1}$; $f_{j,j+1} = f_{j+1,j} = \lambda_j P_j$.

With the modified f_{ij}, the tail distributions $\tilde{H}_j(t) \stackrel{d}{=} \Pr(R_j > t)$ satisfy

$j=1$ $$-P_1\,\tilde{H}_1'(t) + \lambda_1 P_1\,\tilde{H}_1(t) = (\lambda_0 P_0 + \lambda_1 P_1)\,\tilde{B}_1(t) \tag{5.3a}$$

$1<j<K$ $$-P_j\,\tilde{H}_{j-1}'(t) + \lambda_j P_j\,\tilde{H}_j(t) = \lambda_{j-1} P_{j-1}\,\tilde{H}_{j-1}(t) + \lambda_j P_j\,\tilde{B}_j(t) \tag{5.3b}$$

$j=K$ $$-P_K\,\tilde{H}_K'(t) = \lambda_{K-1} P_{K-1}\,\tilde{H}_{K-1}(t) \tag{5.3c}$$

We can compute all $\tilde{H}_j(t)$ for $j \geq 1$. Defining $(N,R) = (0,0)$ for $N=0$, yields

$$\psi(N,R) = P_0\,\psi(0,0) + P_1\,\psi(1,R_1) + P_2\,\psi(2,R_2) + \; + P_K\,\psi(K,R_K) \tag{4.4}$$

We cannot from the foregoing find (certainly not easily) the load or delay or any such variable because at any instant these durations depend also on future arrivals.

Appendix Joint Treatment of System Size and Residual Service In M/G/1

We derive now a single global omni-equation equivalent to equations (1.2) and (1.3).

Definition $\psi N,Z) \overset{d}{=} \psi(0,x)$ when $N=0$ and $\psi N,Z) \overset{d}{=} \psi(N_*,R)$ when $N \geq 1$; $\psi(N,Z)$ is defined at any time point.

Let N_*=system size provided $N \geq 1$. R is defined only when $N \geq 1$ so it needs no asterisk. We assume that $\psi(N,Z)$ is a balanced r.v., so that $Ed\psi(N,Z)=0$. Of course we have

$$\psi(N,Z) = P_0\psi(0,x)+P_*\psi(N_*,R) \quad P_0 = \lambda\bar{x} \quad P_* = 1-\lambda\bar{x} \tag{A.1}$$

Let us consider the balance of $\psi(N,Z)$ during a random dt:

(a) aging, which goes on only while $N \geq 1$, adds

$$Ed\psi(N,Z)|_{aging} = P_* E[\psi(N_*,R-dt) - \psi(N_*,R)] = -dt\, P_* ED_2\psi(N_*,R); \quad P_* = Pr(N \geq 1)$$

(b) arrivals "$0 \to 1$" add $Ed\psi(N,Z)|_{0 \to 1} = f_{01} E[\psi(1,x) - \psi(0,x)]$; $f_{01} = \lambda P_0$

(c) arrivals while $N \geq 1$ add $Ed\psi(N,Z)|_{a*} = f_* E[\psi(N_{a*}+1,R_a) - \psi(N_{a*},R_a)]$; $f_* = \lambda P_*$

A subscript $a*$ says that the r.v. is found by an arrival into a busy system.

Since true poissonian arrivals see, stochastically, what a continuous or random (poissonian) observer sees (cf. Wolff 1982), we have $\psi(N_{a*},R_a) = \psi(N_*,R)$ and

$$f_* E[\psi(N_{a*}+1,R_a) - \psi(N_{a*},R_a)] = f_* E[\psi(N_*+1,R) - \psi(N_*,R)]; \quad f_* = \lambda P_*$$

(d) departures (perforce from a busy system) add

$$Ed\psi(N,Z)|_d = \lambda E[\psi(N_d,x) - \psi(N_d+1,0)]; \text{ departure rate} = \text{arrival rate} = \lambda$$

Subscripts d say that a r.v. is seen by a just departed customer. (System size is N_d+1 just before a departure, and is N_d just after.) Since $\psi(N_d) = \psi(N_a) = \psi(N)$, and since N_d and x are independent,

$$Ed\psi(N,Z)|_d = \lambda E[\psi(N_d,x) - \psi(N_d+1,0)] = \lambda E[\psi(N,x) - \psi(N+1,0)]$$

From (a) through (d) we get (mind the omni-convention!)

$$-P_{\bullet} D_2 \psi(N_{\bullet},R) + f_{01}[\psi(1,x) - \psi(0,x)] + f_{\bullet}[\psi(N_{\bullet}+1,R) - \psi(N_{\bullet},R)] +$$
$$+ \lambda[N,x) - \psi(N+1,0)] = 0 \tag{A.2}$$

where $P_{\bullet} = 1 - P_0 = \rho = \lambda\bar{x}$; $f_{01} = \lambda P_0 = \lambda(1-\rho)$; and $f_{\bullet}$= arrival rate while server works $= \lambda P_{\bullet}$.

With $\psi(N) = P_0\psi(0) + P_{\bullet}\psi(N_{\bullet})$ we get from (A.2) the equation

$$P_{\bullet} D_2 \psi(N_{\bullet},R) - \lambda P_{\bullet}[\psi(N_{\bullet}+1,R) - \psi(N_{\bullet},R)] +$$
$$+ \lambda P_{\bullet}[\psi(N_{\bullet}+1,0) - \psi(N_{\bullet},x)] = \lambda P_0[\psi(1,x) - \psi(1,0)] \tag{A.4}$$

Equation (A.4) does not depend on the definition $Z = x$ when $N = 0$.

By specializing (A.4) to

$$\psi(N_{\bullet},R) = \Pr(N_{\bullet}=j,\ R>t) = P_{\bullet j}\,\tilde{H}_j(t) = P_j\,\tilde{H}_j(t)\big/P_{\bullet} \tag{A.5}$$

where $P_{\bullet j}$ = Pr(system size = j|server works) $= P_j\big/P_{\bullet}$ we get equations (1.4) = (A.6)

j = 1 $$\boxed{-P_1\tilde{H}_1'(t) + \lambda P_1\tilde{H}_1(t) = (\lambda P_0 + \lambda P_1)\tilde{B}(t)} \tag{1.4a)=(A.6a}$$

j ≥ 2 $$\boxed{-P_j\ \tilde{H}_j'(t) + \lambda P_j\tilde{H}_j(t) = \lambda P_{j-1}\tilde{H}_{j-1}(t) + \lambda P_j\tilde{B}(t)} \tag{1.4b)=(A.6b}$$

Equation (A.4) is equivalent to the infinite set (1.2) and (1.3). But (A.4), unlike (A.6), cannot be adapted, we think, to variants of M/G/1.

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